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GRAVITY GRADIENT TORQUES

ON A SUN-ORIENTED SPACE STATION

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ABSTRACT

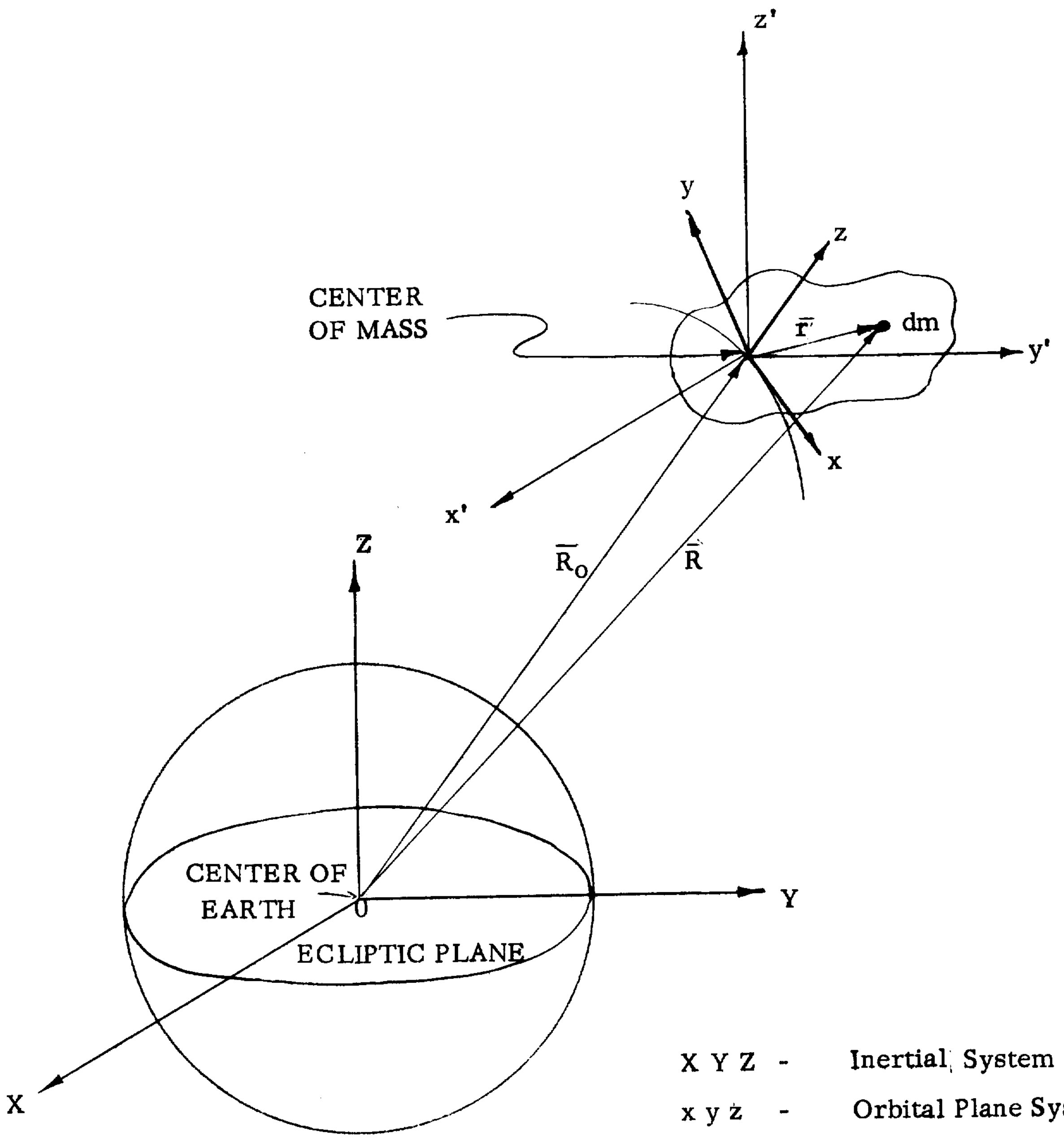
An analysis is presented to determine the time variation of perturbing torques on a space station due to gravity gradient. It is shown that the integrated angular impulse due to gravity gradient on a typical space station exceeds the angular impulse required to precess a spinning station at a rate of 360° per year as required to maintain sun-orientation. If pulse rockets with a specific impulse of 300 seconds were used, the propellant requirement is estimated to be 5880 lb/yr for gravity gradient as compared to 1350 lb/yr for precession.

I ANALYSIS

We consider a space station with a major principal moment of inertia, $I_{z'}$, and equal minor principal moments of inertia $I_{x'}$ and $I_{y'}$, which is continually stabilized so that its major principal axis of inertia points toward the sun. The x' axis is perpendicular to the ecliptic plane and the y' axis lies in the ecliptic plane. For the moment we may neglect the spin of the station about the z' axis and calculate the gravity gradient torques acting on the station as a function of time. (The spin of the station must be considered to determine the angular motions of the station in inertial space if the disturbing gravity gradient torques are not continually counteracted, and also to determine the magnitude and direction of the applied torque required to precess the spin axis 2π radians per year in inertial space to maintain sun-orientation).

The origin of the x', y', z' coordinate system is taken as the center of gravity of the earth. As shown in Figure 1, we let $\bar{R}_o$ be the vector from the center of the earth to the center of mass of the station, $\bar{R}$, the vector from the center of the earth to an elemental mass element, dm, of the station, and $\bar{r}$, the vector from the center of mass of the station to the element, dm. Then the gravity force on dm is:

$$d\,\bar{F}_{dm} = dm\,\bar{a} = -\frac{GM}{R^2}\,dm\,\frac{\bar{R}}{R} \tag{1}$$



X Y Z - Inertial System

x y z - Orbital Plane System

x' y' z' - Body Axes (principal axes)

FIGURE 1

Where	G	=	universal constant of gravitation
	M	=	mass of the earth

or,

$$GM = g_o R_o^2$$

where

g_o	=	acceleration due to gravity at earth's mean surface
R_o	=	earth's mean radius = 3960 statute miles.

The torque about the station center of mass due to the gravity force $d\vec{F}_{dm}$ is:

$$dJ = \bar{r} \times d\bar{F}_{dm} = -\frac{GM\,dm}{R^3}(\bar{r} \times \bar{R}) \tag{2}$$

where $\bar{r}$ is the vector from the center of mass of the station to the element, dm, in a rotating orthogonal coordinate system x y z, with its origin at the station center of mass and defined so that z is radial outward along the earth radius vector, x is perpendicular to the orbit plane and y is in the orbit plane.

But,

$$\bar{R} = \bar{R}_o + \bar{r} \tag{3}$$

The quantity $(1/R^3)$ can be found from the law of cosines:

$$R^2 = r^2 + R_o^2 - 2\bar{r} \cdot \bar{R}_o \tag{4}$$

or,

$$R^3 = (R_o^2 - 2\bar{r} \cdot \bar{R}_o + r^2)^{3/2} \tag{5}$$

$$R^3 = R_o^{\ 3} \left(1 - \frac{2\,\bar{r} \cdot \bar{R}_o}{R_o^{\ 2}} + \frac{r^2}{R_o^{\ 2}} \right)^{3/2} \tag{6}$$

and

$$\frac{1}{R^3} = \frac{1}{R_o^{\ 3}} \left(1 - \frac{2\,\bar{r} \cdot \bar{R}_o}{R_o^{\ 2}} + \frac{r^2}{R_o^{\ 2}} \right)^{-3/2} \tag{7}$$

Since $\frac{r}{R_o} << 1$, we expand the expression by the binomial theorem and retain only the first two terms:

$$\frac{1}{R^3} \cong \frac{1}{R_o^{\ 3}} \left(1 + \frac{3\,\bar{r} \cdot \bar{R}_o}{R_o^{\ 2}} + \cdots \right) \tag{8}$$

Substituting in Equation (2) and replacing $\bar{R}$ by $\bar{R}_o$:

$$d\,\bar{J} = - GM\, dm\, \bar{r} \times \bar{R}_o \left[\frac{1}{R_o^{\ 3}} \left(1 + \frac{3\,\bar{r} \cdot \bar{R}_o}{R_o^{\ 2}}\right) \right] \tag{9}$$

or,

$$d\,\bar{J} = - \frac{GM\, dm}{R_o^{\ 3}} \left[R_o \left(\bar{r} \times \frac{\bar{R}_o}{R_o}\right) + 3\,\bar{r} \cdot \frac{\bar{R}_o}{R_o}\, \bar{r} \times \frac{\bar{R}_o}{R_o} \right] \tag{10}$$

The vector $\frac{\bar{R}_o}{R_o} = \bar{k}$, is a unit vector along the z axis. Therefore, Equation (10) may be written in integral form:

$$\bar{J} = \frac{GM}{R_o^{\ 3}} \left[R_o \int_{Vol} \bar{r} \times \bar{k}\, dm + 3 \int_{Vol} \bar{r} \cdot \bar{k}\ \bar{r} \times \bar{k}\, dm \right] \tag{11}$$

where the integrals are taken over the volume of the station. Since the origin is taken at the center of mass, the first term integrates to zero.

We now express $\bar{k}$ with respect to the body axes x' y' z' :

$$\bar{k} = \cos(\bar{k} \cdot \bar{i}') \cdot \bar{i}' + \cos(\bar{k} \cdot \bar{j}')\, \bar{j}' + \cos(\bar{k} \cdot \bar{k}')\, k' \qquad (12)$$

or,

$$\bar{k} = \cos\theta_1\, \bar{i}' + \cos\theta_2\, \bar{j}' + \cos\theta_3 \bar{k}' \qquad (13)$$

where

$$\bar{r} = x'\, \bar{i}' + y'\, \bar{j}' + z'\, \bar{k}' \qquad (14)$$

and the angles are defined as follows:

θ_1 = angle between earth radius vector and body x' axis

θ_2 = angle between earth radius vector and body y' axis

θ_3 = angle between earth radius vector and body z' axis

Inserting the values of $\bar{r}$ and $\bar{k}$ in Equation (11), and introducing the assumption that x' y' and z' are the principal axes of inertia:

$$\begin{aligned} \bar{J} = & -\frac{3\,GM}{R_o^3} \Big[\Big(\int x'^2\, dm - \int y'^2\, dm \Big) \cos\theta_1 \cos\theta_2 \Big] \bar{k}' \\ & + \Big[\Big(\int z'^2\, dm - \int x'^2\, dm \Big) \cos\theta_1 \cos\theta_3 \Big] \bar{j}' \\ & + \Big[\Big(\int y'^2\, dm - \int x'^2\, dm \Big) \cos\theta_2 \cos\theta_3 \Big] \bar{i}' \end{aligned} \qquad (15)$$

or,

$$J_{x'} = -\frac{3\,GM}{R_o^3}\left[I_y - I_z\right]\cos\theta_2\,\cos\theta_3 \tag{16}$$

$$J_{y'} = -\frac{3\,GM}{R_o^3}\left[I_z - I_x\right]\cos\theta_1\,\cos\theta_3 \tag{17}$$

$$J_{z'} = -\frac{3\,GM}{R_o^3}\left[I_x - I_y\right]\cos\theta_1\,\cos\theta_2 \tag{18}$$

where I_x, I_y and I_z are the principal moments of inertia. Since we have assumed that $I_{x'} = I_{y'}$, Equation (18) shows that there is no gravity gradient torque around the z' (spin) axis which would tend to change the spin rate.

We now refer to Figure 2 which shows the relationship between the satellite orbit plane, the earth's equatorial plane, and the ecliptic plane. Coordinates must be transformed to express the angles θ_1, θ_2, θ_3 as functions of time, t, taking into account the earth's rotation around the sun, the inclination of the earth's axis to the ecliptic plane, the inclination of the orbit to the equatorial plane, and the regression of the line of nodes of the orbit due to the earth's oblateness.

We define:

i	=	the angle of inclination of the orbital plane to the ecliptic plane
i_e	=	orbit inclination to the equatorial plane (30^o)
ψ	=	the angle between the spin axis (z' which points toward the sun) and the intersection of the orbital plane with the ecliptic plane
γ	=	angle between the line of nodes and the vernal equinox (in the equatorial plane).

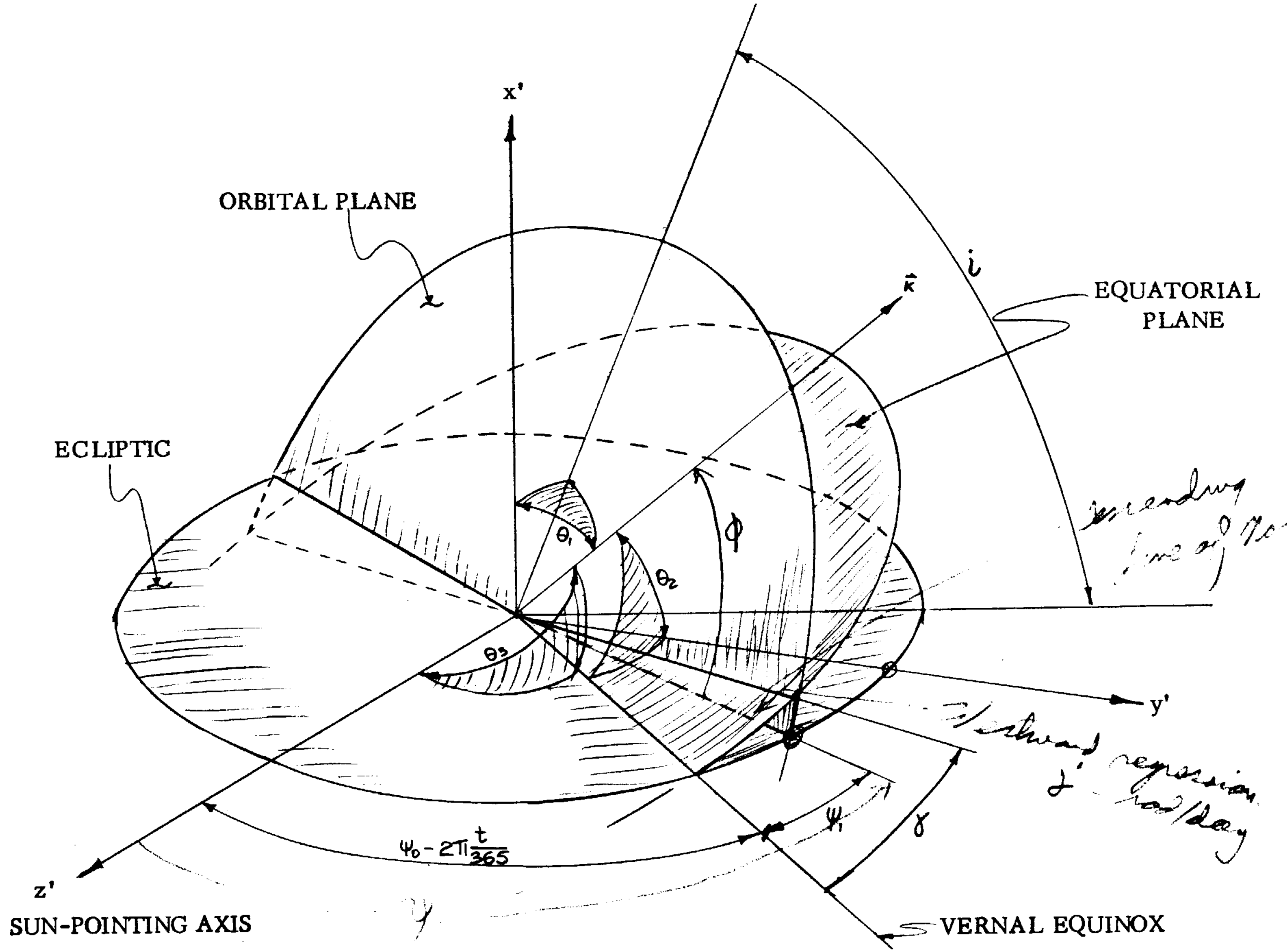

ORBITAL PLANE
EQUATORIAL PLANE
ECLIPTIC
x'
y'
z'
SUN-POINTING AXIS
VERNAL EQUINOX
$\psi_0 - 2\pi\frac{t}{365}$

FIGURE 2

$$\gamma = \gamma_o + \dot{\gamma}\, t \tag{19}$$

where

γ_o = value of γ for the first orbit after launching and $\dot{\gamma}$ is the orbital regression rate due to earth's oblateness.

$$\dot{\gamma} = 2\pi \frac{t}{P_R}$$

where

P_R = regression period

We let:

ψ_o = angle between z' axis and vernal equinox on the date of launching

$$\psi_2 = \psi_o - 2\pi \frac{t}{365} \tag{20}$$

= angle between z' axis and vernal equinox at time, t, (days) after launching

ψ_1 = angle between vernal equinox and the intersection of the orbital plane with the ecliptic plane

$$\Psi = \psi_1 + \psi_2 = \left[\psi_1 + \psi_o - 2\pi \frac{t}{365} \right]$$

ϕ = angular position of satellite in its orbit, measured from the ascending line of nodes.

To obtain an expression for the variation of torque with time, it is necessary to define the angles θ_1, θ_2, and θ_3 of the Equations (16) and (17) in terms of ψ and γ. This will be accomplished in two steps. First, applying the cosine law to the spherical triangles indicated in Figure 2, the following relationships are obtained for $\cos\theta_1$, $\cos\theta_2$, and $\cos\theta_3$ as a function of the angles ϕ, ψ, and i:

$$\cos\theta_1 = \sin i \sin\phi + \cos i \cos\phi \cos 90^{\circ} \tag{22}$$

$$\cos\theta_2 = \cos\phi \sin\psi + \sin\phi \cos\psi \cos i \tag{23}$$

$$\cos\theta_3 = \cos\psi \cos\phi + \sin\psi \sin\phi \cos(180 - i) \tag{24}$$

Substituting these relations in Equations (16 and (17):

$$J_{x'} = -\frac{3\,GM}{R_o^3}(I_{z'} - I_{y'})\left\{\frac{1}{2}\left[\cos i \quad (\cos^2\psi - \sin^2\psi)\sin 2\phi + \sin 2\psi\,(\cos^2\phi - \cos^2 i \sin^2\phi)\right]\right\} \tag{25}$$

$$J_{y'} = +\frac{3\,GM}{R_o^3}(I_{z'} - I_{x'})\left\{\frac{1}{2}\left[\sin i \cos\psi \sin 2\phi - \sin 2i \sin^2\phi \sin\psi\right]\right\} \tag{26}$$

The first term in each of the above expressions contains the factor $\sin 2\phi$. These torque components will therefore be periodic at twice the orbital frequency. Since the orbital period (of the order of 90 minutes) is much shorter than either the earth's precession period about the sun (365 days) or the orbit regression period

(47 days), both i and ψ are essentially constant over half an orbital period. Hence, this component of torque averages to zero and may be counteracted by a momentum-wheel or precessing gyro type of control device. However, the second term in Equations (25) and(26) contains only $\sin^2 \phi$ and $\cos^2 \phi$. Over one orbital period:

$$(\sin^2 \phi)_{average} = (\cos^2 \phi)_{average} = \frac{1}{2}$$

The cumulative gravity-gradient torques are therefore:

$$J_{x'} = - \frac{3\,GM}{4\,R_o^3} (I_{z'} - I_{y'}) \sin^2 i \sin 2\,\psi \tag{26}$$

$$J_{y'} = - \frac{3\,GM}{4\,R_o^3} (I_{z'} - I_{x'}) \sin 2\,i \sin \psi \tag{27}$$

The variation of i and ψ with time is obtained from the solution of the spherical triangle shown in Figure 3. The following relationships are obtained:

$$\cos i = 0.917 \cos i_{eq} - 0.398 \sin i_{eq} \cos \gamma \tag{28}$$

$$\tan \psi_1 = \frac{\tan \gamma}{\left[0.917 + \frac{(0.398)\ \cot\ (i_{eq})}{\cos \gamma} \right.} \tag{29}$$

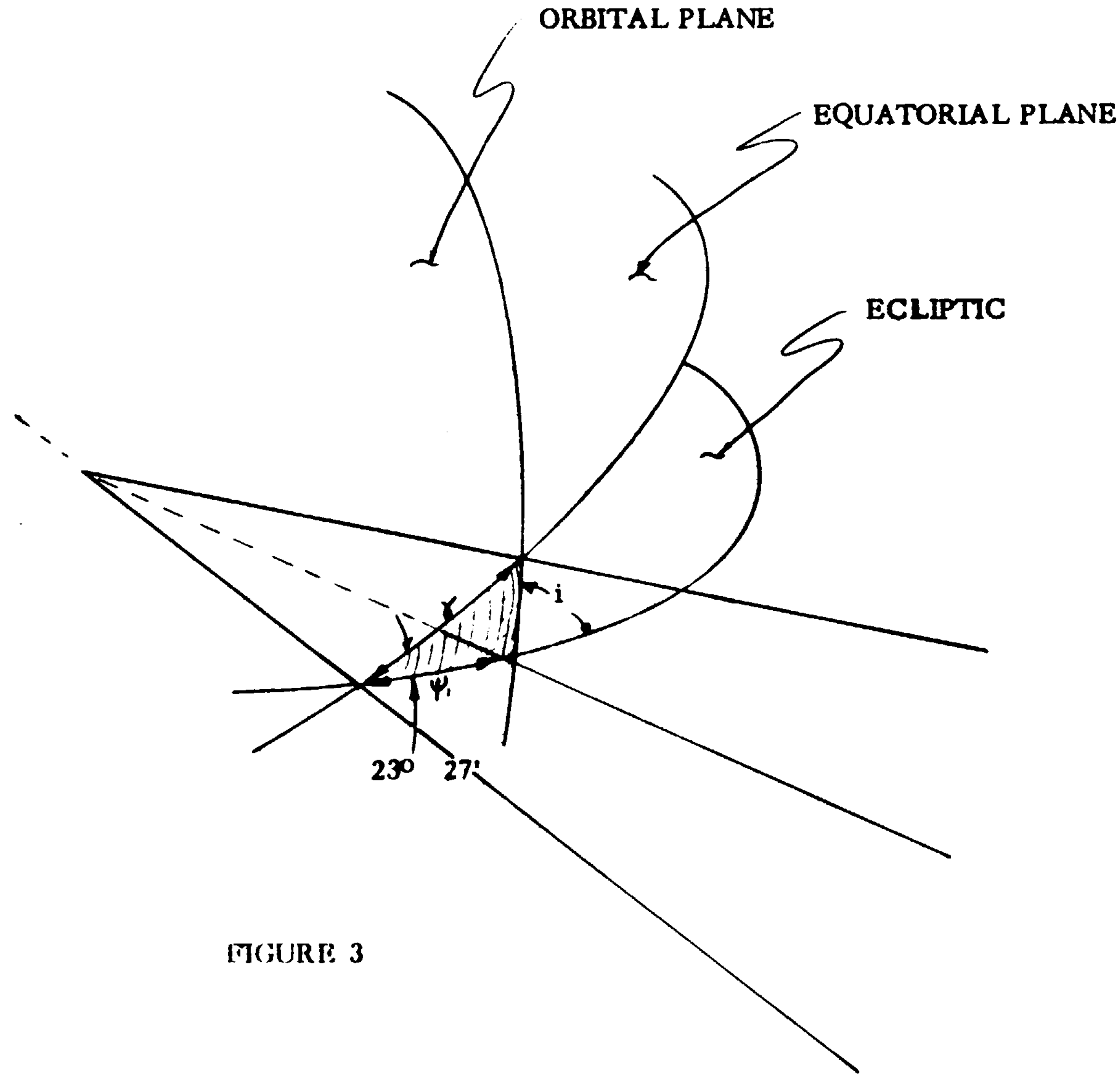


FIGURE 3

As previously defined, the angle γ consists of an initial value γ_o and a periodic component due to regression of the orbital plane $\dot{\gamma}$. In general, the regression rate for a circular orbit is given by:

$$\dot{\gamma} = \frac{6}{5} \frac{(GM)^{1/2} R_e^{\ 2}}{R_o^{\ 7/2}} \left[(0.0032) \cos i_{eq} \sin^2 \phi \right] \tag{30}$$

where

R_e = earth's radius

(Reference: Thomson, Introduction to Space Dynamics, page 98)

As previously noted, the orbital period is much smaller than the orbit regression period $2\pi/\dot{\gamma}$ and we may again take:

$$\sin^2 \phi = \frac{1}{2}$$

Therefore:

$$\dot{\gamma} = (0.00192) \frac{(GM)^{1/2} R_e^{\ 2}}{R_o^{\ 7/2}} \cos i_{eq} \tag{31}$$

To obtain an explicit expression for the variation of gravity-gradient torques with time it would be necessary to combine Equations (26) and (27) with Equations (28), (29) and (31). Also,

$$\psi = \psi_o - 2\pi \frac{t}{365} + \psi_1 \tag{32}$$

Since such an explicit expression would only obscure the result, the evaluation is best done in a step-wise manner. Specifically, it is most instructive to consider the variation of $\sin^2 i$ and $\sin 2 i$ separately from the terms $\sin 2 \psi$ and $\sin \psi$ in Equations (26) and (27). As will be shown below, the first influence only the magnitude of the torques, whereas the effect of the latter is to establish their fundamental period.

From the geometry of the problem, and as is also shown by Equation (28), the angle of the orbital plane to the ecliptic, i, will vary between the limits:

$$i_{eq} - 23.5^{o} < i < i_{eq} + 23.5^{o} \tag{32}$$

Since for the orbits to be considered here, the inclination will be larger than 23.5^{o}, the angle i is positive and both $\sin^2 i$ as well as $\sin 2 i$ will always be positive. Hence, these terms are positive multipliers of the torque magnitude. The numerical value of these multipliers changes with time and the variation is periodic at a period given by $2\pi / \dot{\gamma}$ and $2 (2 \pi / \dot{\gamma})$.

The components of ψ given by Equations (32) and (29) are illustrated in Figure 4 (a). The angle ψ_1 would be equal to γ except for the distortion introduced by the denominator of Equation (29). However, the distortion is periodic with a period also given by $2 \pi / \dot{\gamma}$. The total angle ψ therefore has a fundamental period P_ψ , as shown in Figure 4 (b).

$$P_\psi = \frac{P_r (365)}{365 - P_r} \tag{34}$$

where

$$P_\gamma = 2 \pi / \dot{\gamma} \text{ days}$$

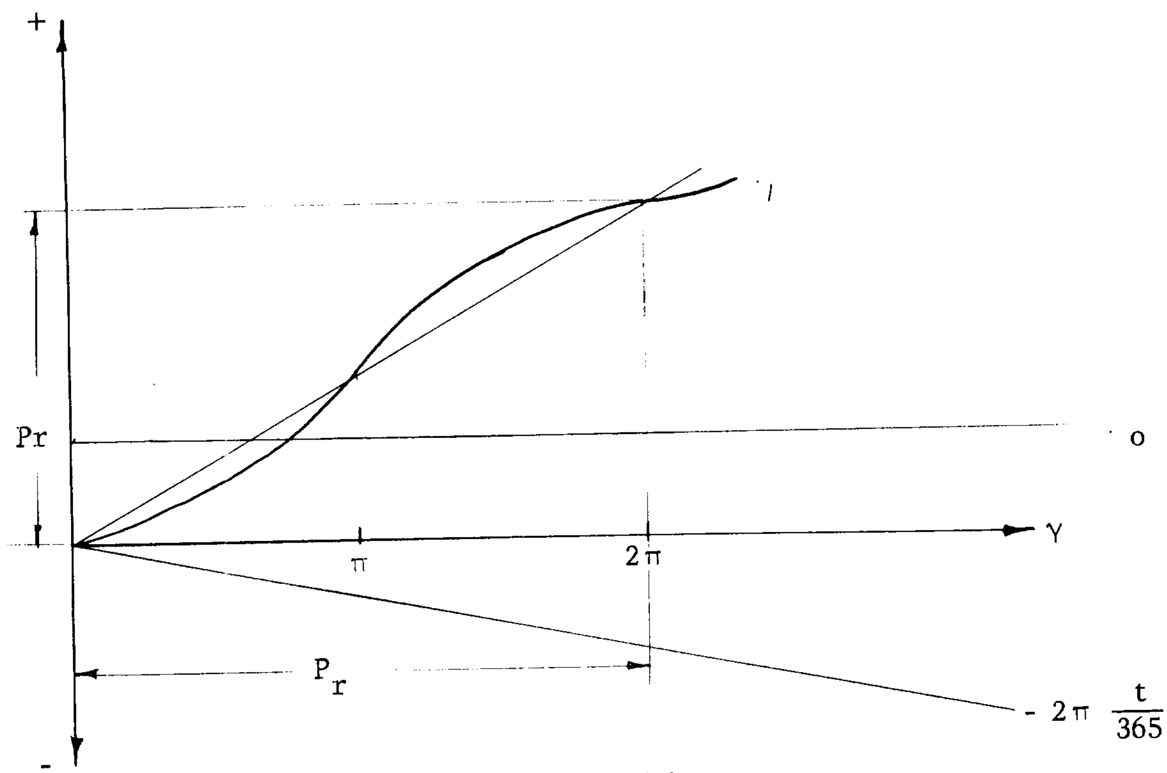


FIGURE 4 (a)

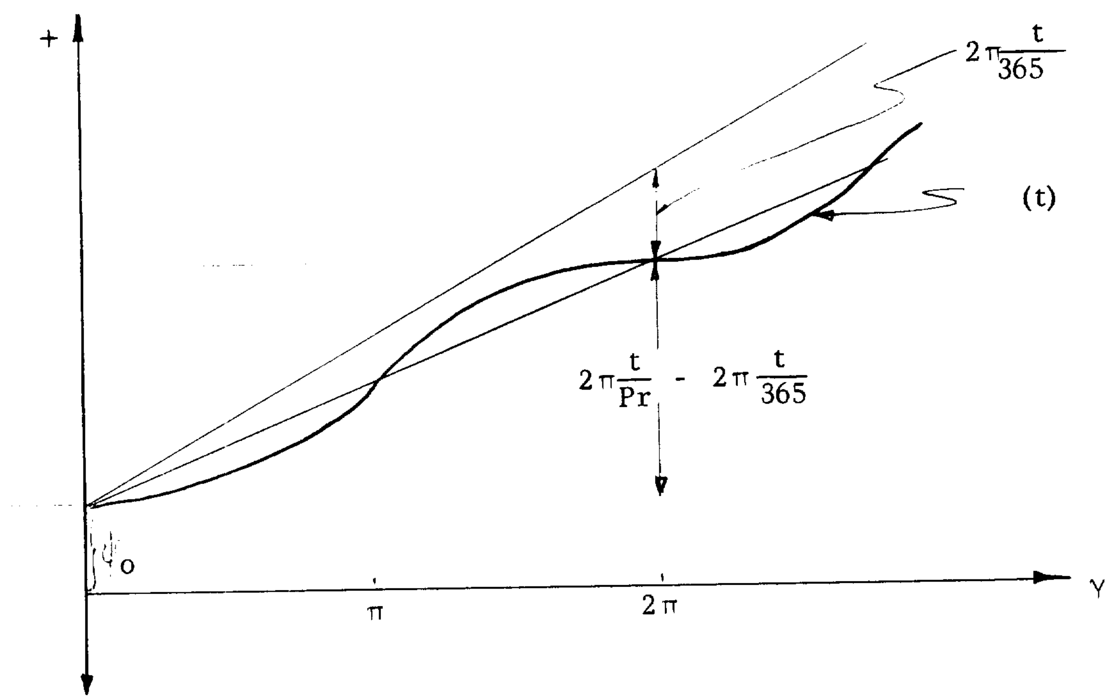


FIGURE 4 (a)

Assuming that the cumulative gravity gradient torques are continuously counteracted by jet-reaction torques, the estimate of fuel requirements is made by evaluating the area under the curves of $J_{x'}(t)$ and $J_{y'}(t)$ without regard to sign. Since the distortion introduced by the ψ_1 term is approximately symmetrical over each half period, its effect upon the absolute value of the area under the curve taken over a full period is negligible for estimating purposes. Hence, the principal concern must be with the influence of the $\sin^2 i$ and $\sin 2 i$ terms. The torque equations can therefore be written as:

$$J_{x'} = - \frac{3 \, GM}{4 \, R_o^3} (I_{z'} - I_{y'}) \sin^2 i \sin 2 (\psi_o + \gamma_o + 2 \pi \frac{t}{P_\psi}) \tag{35}$$

$$J_{y'} = - \frac{3 \, GM}{4 \, R_o^3} (I_{z'} - I_{x'}) \sin 2 i \sin (\psi_o + \gamma_o + 2 \pi \frac{t}{P_\psi}) \tag{36}$$

II NUMERICAL ESTIMATES

The particular case for which cumulative gravity-gradient torques and the corresponding fuel requirements have been estimated assumes the following orbit and vehicle parameters:

Orbit inclination to equator, i_{eq}	=	30^o
Circular orbit altitude	=	300 nautical miles
Spin-axis moment of inertia, $I_{z'}$	=	1.5×10^7 slug - ft^2
$I_{x'}$	=	$I_{y'} = 1.05 \times 10^7$ slug - ft^2

From Equation (30), the orbit regression rate $\dot{\gamma}$ and regression period P_γ are:

$$\dot{\gamma} = 1.54 \times 10^{-6} \text{ rad/sec}$$

$$P_\gamma = 47.3 \text{ days}$$

The fundamental period of the cumulative gravity-gradient torques is computed from Equation (34).

$$P_\psi = 54.3 \text{ days}$$

The cumulative torque expressions become:

$$J_{x'} = -4.06 \cdot \sin^2 i \cdot \sin 2 (\psi_o + \gamma_o + 2 \pi \frac{t}{54.3} \qquad (37)$$

$$J_{y'} = -4.06 \cdot \sin 2 i \cdot \sin (\psi_o + \gamma_o + 2 \pi \frac{t}{54.3} \qquad (38)$$

The principal difficulty in assigning a numerical value to $\sin^2 i$ and $\sin 2i$ stems from the fact that (1) their period differs from the fundamental period of ψ_1 by a factor of about 2 (47.3 days compared with 27.2 days for J_x, and 23.6 days as compared with 54.3 days for J_y,); and (2) they are displaced by a phase angle determined by the date and time-of-day of injection into orbit. A typical curve is illustrated in Figure 5, based upon an arbitrary choice of phase angles.

To estimate the extent to which the $\sin^2 i$ and $\sin 2i$ terms will reduce the area under the curve, i.e., the accumulated angular momentum, these functions are plotted in figure 6. It is thus seen that appropriate average values for these functions might be:

$$(\sin^2 i)_{average} \approx 0.4$$

$$(\sin 2i)_{average} \approx 0.8$$

It is now possible to estimate fuel requirements for jet-reaction control of the cumulative gravity gradient torques. We assume the following:

Nozzle moment arm	-	75 feet
Specific impulse	-	300 seconds (for chemical rocket)

Hence, the weight of fuel, W, is given by:

$$W = 4.44 \times 10^{-5} \text{ H lbs}$$

where H is the accumulated angular momentum over the specified period of time (lb-ft-sec).

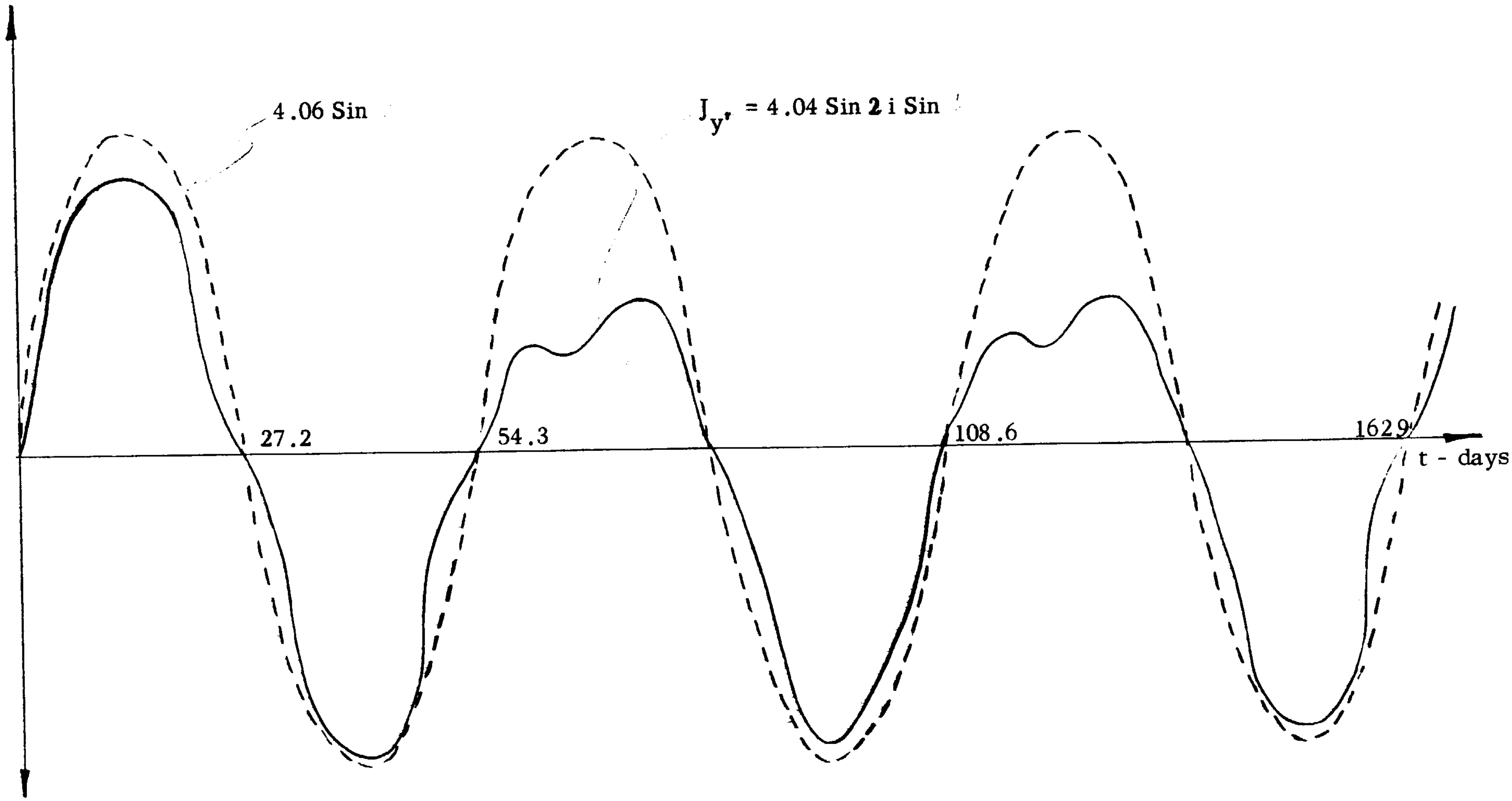


FIGURE 5

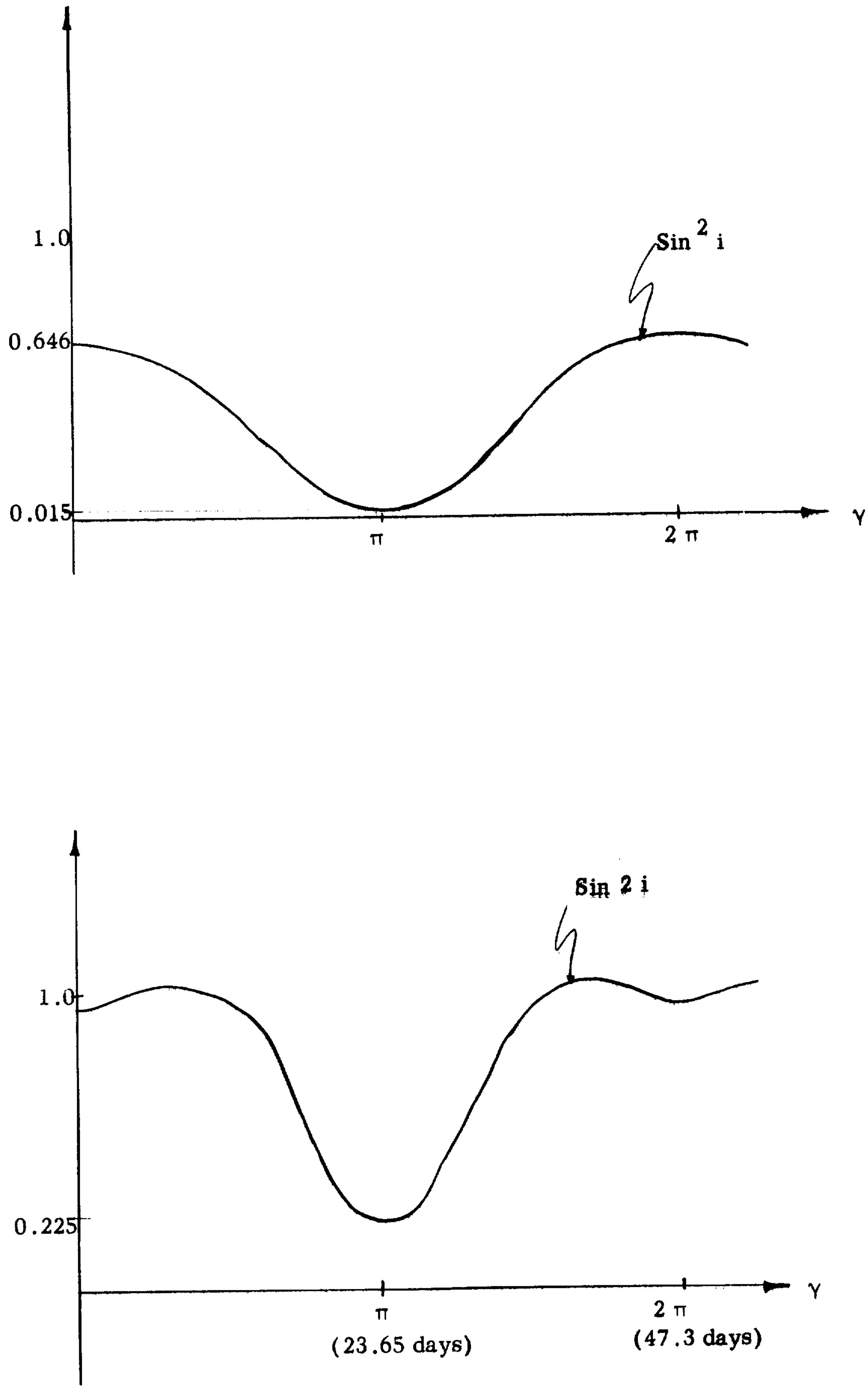


FIGURE 6

Using the above average values for the functions of i, fuel requirements per year are calculated from:

$$W_{x'} = (4.44 \times 10^{-5}) \times (4.06)(0.4) \int_0^{13.6 \text{ days}} \sin 2\pi \frac{t}{27.2} \, dt \times (8.64)(10^4)\left(\frac{365}{13.6}\right)$$

$$W_{x'} = 1{,}455 \text{ lbs per year}$$

$$W_{y'} = (4.44 \times 10^{-5})(4.06)(0.8) \int_0^{27.2 \text{ days}} \sin 2\pi \frac{t}{54.3} \, dt (8.64)(10^4)\left(\frac{365}{27.2}\right)$$

$$W_{y'} = 4{,}425 \text{ lbs per year}$$

If the x and y torques were compensated independently, the total propellant consumption (for a specific impulse of 300 seconds) would be 5,880 lb/year.

For the assumed space station parameters and with a spin rate of 3 RPM, the average torque required to precess the station at a rate of 360° per year is 0.963 lb-ft.

The amount of chemical propellant required to produce this precession is calculated to be only 1350 lb/year, as compared to approximately 5880 lb/year to compensate for gravity gradient.